Inequalities for Partitioned Positive Semidefinite Matrices*

JOHN DE PILLIS
University of California
Riverside, California

Communicated by Alan J. Hoffman

1. INTRODUCTION

In a paper by R. C. Thompson [6] it was proved that, for the positive definite $mn \times mn$ matrix $P = (A_{ij})$, A_{ij} is an $n \times n$ matrix, $i, j = 1, 2, \ldots, m$, we have

$$\det(P) = \det\begin{bmatrix} A_{11} & \ldots & A_{1m} \\ & \vdots & \\ A_{m1} & \ldots & A_{mm} \end{bmatrix} \leqslant \det\begin{bmatrix} \det(A_{11}) & \ldots & \det(A_{1m}) \\ & \vdots & \\ \det(A_{m1}) & \ldots & \det(A_{mm}) \end{bmatrix}, \tag{1.1}$$

where equality obtains if and only if P is block diagonal, i.e., if and only if $A_{ij} = 0$, the zero $n \times n$ matrix whenever $i \neq j$. Under this condition, then, we have $\det(P) = \det(A_{11})\det(A_{22}) \cdots \det(A_{mm})$. Since the advent of Thompson's paper, various generalizations of (1.1) have appeared. For example, M. Marcus [3, Th. 3] refined (1.1) by showing that

$$\underset{i,j=1,2,\ldots,k,k+1}{\frac{\det((A_{ij})_{ij})}{\det((\det(A_{ij}))_{ij})}} \leqslant \underset{i,j=1,2,\ldots,k}{\frac{\det((A_{ij})_{ij})}{\det((\det(A_{ij}))_{ij})}} \leqslant \cdots \leqslant \underset{i,j=1}{\frac{\det(A_{11})}{\det(\det(A_{11}))}} = 1 \tag{1.2}$$

* The research reported was partially supported by NSF GP 5262. The paper was delivered at the Second Symposium on Inequalities, August 1967, at Colorado Springs, Colorado.

for all $k = 1, 2, \ldots, m-1$; we have used the notation

$$(a_{ij})_{ij}$$

to denote the matrix whose ijth entry is the quantity (scalar or matrix) a_{ij}. This result (1.2) of Marcus' follows, in turn, from certain results of de Pillis [1, cf. Th. 3.1] concerning the Hilbert space structure of Grassmann algebras. In the present paper we seek to extend (1.1) to the general elementary symmetric functions $E_1 =$ trace, $E_2, \ldots, E_{r-1}$, $E_r =$ determinant, on $r \times r$ matrices. More specifically,

DEFINITION. $E_q(\cdot)$, the qth *elementary symmetric function*, is defined on the $r \times r$ matrix A by setting $E_q(A)$ equal to the $\binom{r}{q}$-term sum of the determinants of *all* of the principal $q \times q$ submatrices of A. (Precise definitions follow in Section 2; see also [4, p. 17].) That is, for each $q = 1, 2, \ldots, r$, and for each $r \times r$ matrix A, we write

$$E_q(A) = \sum_{\sigma} \det(A[\sigma|\sigma]), \tag{1.3}$$

where $A[\sigma|\sigma]$ represents the $q \times q$ principal submatrix of A based on the $\sigma(1)$th, $\sigma(2)$th, $\ldots$, $\sigma(q)$th rows and columns of A, and σ runs over the $\binom{r}{q}$-element set of integer-valued functions on $\{1, 2, \ldots, q\}$ with the constraint that $1 \leqslant \sigma(1) < \sigma(2) < \cdots < \sigma(q) \leqslant r$.

Our objective is to compare the quantities

$$E_{pq}\begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ & & \vdots & \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix} \tag{1.4a}$$

and

$$E_{p}\begin{bmatrix} E_q(A_{11}) & E_q(A_{12}) & \ldots & E_q(A_{1m}) \\ E_q(A_{21}) & E_q(A_{22}) & \ldots & E_q(A_{2m}) \\ & & \vdots & \\ E_q(A_{m1}) & E_q(A_{m2}) & \ldots & E_q(A_{mm}) \end{bmatrix}, \tag{1.4b}$$

where, as before, $P = (A_{ij})_{ij}$ is the $mn \times mn$ positive definite partitioned matrix, whose ijth block entry is the $n \times n$ matrix A_{ij}, $i, j = 1, 2, \ldots, m$. As to the restriction on the integers p and q, we require that $1 \leqslant p \leqslant m$ and $1 \leqslant q \leqslant n$. Observe that, for the maximal values for p and q, i.e., for $p = m$ and $q = n$ (which implies that $E_p = E_q = E_{pq} =$ determinant), we have from (1.1) that quantity (1.4a) is less than or equal to quantity (1.4b). For another example, chose p and q at their minimal values; i.e., $p = q = 1$. Thus $E_p = E_q = E_{pq} =$ trace, and quantity (1.4a) is trivially *equal* to quantity (1.4b).

Unfortunately, we cannot correctly conjecture that $E_{pq}(P)$ in (1.4a) is always less than or equal to (1.4b) for all p, q; more often than not, the opposite is the case. We do, however, show the following (Theorem 4.3):

Let $P = (A_{ij})_{ij}$, $i, j = 1, 2, \ldots, m$, be an $mn \times mn$ scalar-entried, block-partitioned, positive semidefinite matrix (each A_{ij} is an $n \times n$ matrix). Then, for all integers p, q, where $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant n$, we have

$$\sum_{P'} \det(P') \leqslant E_p \begin{bmatrix} E_q(A_{11}) & E_q(A_{12}) & \ldots & E_q(A_{1m}) \\ E_q(A_{21}) & E_q(A_{22}) & \ldots & E_q(A_{2m}) \\ & & \vdots & \\ E_q(A_{m1}) & E_q(A_{m2}) & \ldots & E_q(A_{mm}) \end{bmatrix} \tag{1.5}$$

($P' =$ *uniform* principal $pq \times pq$ submatrix of P), where summation in (1.5) proceeds not over *all* principal $pq \times pq$ submatrices of P (in which case the sum would equal $E_{pq}(P)$; cf. (1.3)), but only over *certain* $pq \times pq$ principal submatrices P' of P, called *uniform principal submatrices* of P, which we now define.

Definition. Let $A = (A_{ij})_{ij}$, $i, j = 1, 2, \ldots, m$, be an $mn \times mn$ scalar-entried, partitioned matrix, where each A_{ij} is an $n \times n$ matrix $(n > 1)$. A principal $pq \times pq$ submatrix A' of A is called a *uniform principal submatrix* of A (relative to the block partitioning of A) if A' is obtained from A by extracting exactly p rows (and corresponding columns) from each of q block rows (and corresponding block columns) of A. All other principal $pq \times pq$ submatrices of A are *nonuniform*.

The schematic representation (1.6) illustrates the mechanics of selecting pq columns (and corresponding rows) to obtain a typical uniform principal submatrix of A; A is partitioned into m block columns and rows, each block column of A consists of n scalar-entried columns. The definition tells us, then, to select first any p (out of m) block columns and corresponding block rows from A. Then, from each of these block columns and rows, select exactly q columns and corresponding rows; the manner in which these q columns (and corresponding rows) are chosen may vary from block column to block column. The resulting $pq \times pq$ principal submatrix thus obtained is a typical uniform principal submatrix of A relative to the partitioning of A.

$$
\begin{array}{l}
\overbrace{\downarrow\ \downarrow \cdots \downarrow}^{q\text{ columns}} \qquad\qquad\qquad \overbrace{\downarrow\ \downarrow \cdots \downarrow}^{q\text{ columns}} \quad \overbrace{\downarrow\ \downarrow \cdots \downarrow}^{q\text{ columns}} \\
|n \text{ entries}|n \text{ entries}|n \text{ entries}|\cdots|n \text{ entries}|\cdots|n \text{ entries}| \\
\quad \underbrace{\uparrow \qquad\qquad\qquad \uparrow \quad \cdots \quad \uparrow}_{p\text{ block columns}} \\
\quad \underbrace{\uparrow \qquad \uparrow \qquad \uparrow \quad \cdots \quad \uparrow \quad \cdots \quad \uparrow}_{m\text{ block columns}}
\end{array}
\tag{1.6}
$$

To round out our result, we show that equality obtains in (1.6) if and only if $P = (A_{ij})_{ij}$ is block diagonal to begin with, i.e., if and only if the $n \times n$ matrices A_{ij} are zero whenever $i \neq j$.

2. PRELIMINARIES AND DEFINITIONS

For vectors $x_1, x_2, \ldots, x_q$ of mn-dimensional unitary space $\mathscr{H}_{mn}$ with inner product $\langle \cdot, \cdot \rangle$, we denote the Grassmann exterior product [5, Chap. 16, Sec. 10] by

$$x_1 \wedge x_2 \wedge \cdots \wedge x_q. \tag{2.1}$$

Vectors of the form (2.1) are called *decomposable* vectors (of degree q). The vector space spanned by all decomposable vectors (2.1) is denoted by the symbol $\wedge^q \mathscr{H}_{mn}$, and its dimension is

$$\binom{mn}{q} = \frac{(mn)!}{(mn-q)!q!}$$

for $1 \leqslant q \leqslant mn$; the dimension of $\wedge^q \mathscr{H}_{mn}$ equals zero for $q > mn$. The space $\wedge^q \mathscr{H}_{mn}$ can be made into a Hilbert space by defining the inner product on pairs of decomposable vectors by

$$\langle x_1 \wedge x_2 \wedge \cdots \wedge x_q, y_1 \wedge y_2 \wedge \cdots \wedge y_q \rangle_q = \det((\langle x_i, x_j \rangle)_{ij}), \tag{2.2}$$

the determinant of the $q \times q$ matrix whose ijth entry is the scalar $\langle x_i, y_j \rangle$, for all $x_i, y_j \in \mathscr{H}_{mn}$. The symbol $Q_{n,q}$, $n \geqslant q$, will denote that class of one-to-one order-preserving functions σ which send the set $\{1, 2, \ldots, q\}$ into $\{1, 2, \ldots, q, \ldots, n\}$. That is, $\sigma \in Q_{n,q}$ if and only if $1 \leqslant i < j \leqslant q \Rightarrow 1 \leqslant \sigma(i) < \sigma(j) \leqslant n$. For the vectors $x_1, x_2, \ldots, x_q, \ldots, x_n \in \mathscr{H}_{mn}$, we define $\mathbf{x}_\sigma$ in $\wedge^q \mathscr{H}_{mn}$ by

$$\mathbf{x}_\sigma = x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(q)}. \tag{2.3}$$

We note, relative to the inner product (2.2) defined for $\wedge^q \mathscr{H}_{mn}$, that, if $\chi_r = \{x_1, \ldots, x_r\}$ is an orthonormal set of $\mathscr{H}_{mn}$, then $\{\mathbf{x}_\sigma : \sigma \in Q_{r,q}\}$ is an $\binom{r}{q}$-element, orthonormal set in $\wedge^q \mathscr{H}_{mn}$. If $r = mn$, so that χ_r is an orthonormal *basis* for $\mathscr{H}_{mn}$, then the $\binom{mn}{q}$-element set $\{\mathbf{x}_\sigma : \sigma \in Q_{mn,q}\}$ is an orthonormal *basis* for $\wedge^q \mathscr{H}_{mn}$.

For any linear operator $\mathbf{T}$ on Hilbert space $\mathscr{H}$, $C_q(\mathbf{T})$, the qth elementary compound of $\mathbf{T}$ is that linear operator defined on $\wedge^q \mathscr{H}$ by

$$C_q(\mathbf{T})(x_1 \wedge x_2 \wedge \cdots \wedge x_q) = \mathbf{T}x_1 \wedge \mathbf{T}x_2 \wedge \cdots \wedge \mathbf{T}x_q \tag{2.4}$$

for all decomposable vectors $x_1 \wedge x_2 \wedge \cdots \wedge x_q \in \wedge^q \mathscr{H}$. From (2.4) we see that $C_q(\mathbf{T} \cdot \mathbf{U}) = C_q(\mathbf{T}) C_q(\mathbf{U})$ for linear operators $\mathbf{T}$ and $\mathbf{U}$ on $\mathscr{H}$. Using (2.4) and (2.2), we deduce that $C_q(\mathbf{T})^* = C_q(\mathbf{T}^*)$.

Now, for any linear operator $\mathbf{T}$ on $\mathscr{H}$, we define $E_q(\mathbf{T})$, the qth elementary symmetric function of $\mathbf{T}$, by

$$E_q(\mathbf{T}) = \operatorname{tr}(C_q(\mathbf{T})), \quad \text{the trace of } C_q(\mathbf{T}). \tag{2.5a}$$

If $m \times \mathbf{T}$ denotes the matrix of $\mathbf{T}$ relative to some ordered basis for $\mathscr{H}$, we define the qth elementary function on $m \times \mathbf{T}$ by

$$E_q(mx\mathbf{T}) = E_q(\mathbf{T}). \tag{2.5b}$$

Note that (2.5b) is well defined in that the definition is independent of basis $(E_q(mx\mathbf{T}) = E_q(S \cdot S^{-1} mx\mathbf{T}) = E_q(S^{-1} \cdot mx\mathbf{T} \cdot S)$ for any invertible square matrix S). An equivalent definition for $E_q(A)$, where A is an $n \times n$ matrix, follows: If $A = (a_{ij})_{ij}$, then $E_q(A)$ is the sum of the determinants of each of the $\binom{n}{q}$ $q \times q$ principal submatrices of A (cf. [4, p. 17]), i.e.,

$$E_q(A) = \sum_{\sigma \in Q_{n,q}} \det \begin{bmatrix} a_{\sigma(1)\sigma(1)} & a_{\sigma(1)\sigma(2)} & \cdots & a_{\sigma(1)\sigma(q)} \\ a_{\sigma(2)\sigma(1)} & a_{\sigma(2)\sigma(2)} & \cdots & a_{\sigma(2)\sigma(q)} \\ & & \vdots & \\ a_{\sigma(q)\sigma(1)} & a_{\sigma(q)\sigma(2)} & \cdots & a_{\sigma(q)\sigma(q)} \end{bmatrix}. \tag{2.5c}$$

If $a_{ij} = \langle x_i, y_j \rangle$ for $x_i, y_j \in \mathscr{H}$, then from (2.2), (2.3), and (2.5c) we have

$$E_q(A) = \sum_{\sigma \in Q_{n,q}} \langle \mathbf{x}_\sigma, \mathbf{y}_\sigma \rangle_q. \tag{2.6}$$

Let us consider now the exterior product of the vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p$, where each $\mathbf{x}^i$ belongs to the Hilbert space $\wedge^q \mathscr{H}$. We use the symbol $\overline{\wedge}$, as distinguished from $\wedge$, the exterior product of vectors in $\mathscr{H}$. Thus we write the decomposable vector as follows:

$$\mathbf{x}^1 \overline{\wedge}\, \mathbf{x}^2 \overline{\wedge} \cdots \overline{\wedge}\, \mathbf{x}^p, \qquad \mathbf{x}^i \in \wedge^q \mathscr{H}. \tag{2.7}$$

The vector space spanned by all decomposable vectors (2.7) is denoted $\overline{\wedge}^p(\wedge^q \mathscr{H})$. Analogous to (2.2), we endow $\overline{\wedge}^p(\wedge^q \mathscr{H})$ with an inner product $\langle\langle \cdot, \cdot \rangle\rangle_p$, which is defined on ordered pairs of decomposable vectors as follows:

$$\langle\langle \mathbf{x}^1 \overline{\wedge} \cdots \overline{\wedge}\, \mathbf{x}^p, \mathbf{y}^1 \overline{\wedge} \cdots \overline{\wedge}\, \mathbf{y}^p \rangle\rangle_p = \det((\langle \mathbf{x}^i, \mathbf{y}^j \rangle_q)_{ij}), \tag{2.8}$$

the determinant of the $p \times p$ matrix whose ijth entry is the scalar $\langle \mathbf{x}^i, \mathbf{y}^j \rangle_q$, $\mathbf{x}^i, \mathbf{y}^j \in \wedge^q \mathscr{H}$. We shall need the following result, which appears as Corollary 2.5 in [1].

LEMMA 2.1. *Suppose* $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p\}$ *is a set of decomposable vectors in* $\wedge^q \mathscr{H}$, *i.e.*, $\mathbf{x}^i = x_1{}^i \wedge x_2{}^i \wedge \cdots \wedge x_q{}^i$ *for each* $i = 1, 2, \ldots, p$. *Then*

$$\langle\langle \mathbf{x}^1 \overline{\wedge} \cdots \overline{\wedge}\, \mathbf{x}^p, \mathbf{x}^1 \overline{\wedge} \cdots \overline{\wedge}\, \mathbf{x}^p \rangle\rangle_p \geqslant \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^p, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^p \rangle_{pq}, \tag{2.9}$$

where equality obtains if and only if each $x_r{}^i$ *is orthogonal to each* $x_s{}^j$, *whenever* $i \neq j$. *Moreover, in the case of equality, both sides of* (2.9) *take on the value*

$$\langle \mathbf{x}^1, \mathbf{x}^1 \rangle_q \langle \mathbf{x}^2, \mathbf{x}^2 \rangle_q \cdots \langle \mathbf{x}^p, \mathbf{x}^p \rangle_q;$$

(equivalently)

$$\langle\langle \mathbf{x}^1, \mathbf{x}^1 \rangle\rangle_1 \langle\langle \mathbf{x}^2, \mathbf{x}^2 \rangle\rangle_1 \cdots \langle\langle \mathbf{x}^p, \mathbf{x}^p \rangle\rangle_1.$$

We shall also need the following fact concerning orthogonal projections on $\mathscr{H}$ (a more general statement relative to partial isometries is to be found in [2, Lemma 3.1]).

LEMMA 2.2. *Let $\mathscr{M}$ denote both the subspace of Hilbert space $\mathscr{H}$ and the orthogonal projection onto that subspace. If $\mathscr{M}$ has orthonormal basis $\{e_1, \ldots, e_k\}$, then*

$$C_k(\mathscr{M}) = (e_1 \wedge e_2 \wedge \cdots \wedge e_k \times e_1 \wedge e_2 \wedge \cdots \wedge e_k), \tag{2.10}$$

the one-dimensional orthogonal projection onto the subspace of $\wedge^k\mathscr{H}$ spanned by the vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$. (Recall that, for vectors $\mathbf{x}, \mathbf{y} \in \wedge^k\mathscr{H}$, the dyad operator, $(\mathbf{x} \times \mathbf{y})$, is defined by $(\mathbf{x} \times \mathbf{y})\mathbf{z} = \langle \mathbf{z}, \mathbf{y}\rangle_k \mathbf{x}$ for all $\mathbf{z} \in \wedge^k\mathscr{H}$.)

We may define an inner product on the algebra of all linear operators on Hilbert space $\mathscr{H}$ by putting

$$[\mathbf{T}, \mathbf{U}] = \operatorname{tr}(\mathbf{U}^*\mathbf{T}), \quad \text{the trace of } \mathbf{U}^*\mathbf{T}, \tag{2.11}$$

for all linear operators $\mathbf{T}$, $\mathbf{U}$, where $\mathbf{U}^*$ is the Hilbert space adjoint of $\mathbf{U}$.

We combine some of these ideas to introduce the following result.

LEMMA 2.3. *Let $\mathscr{P}$ be a linear operator on M-dimensional Hilbert space $\mathscr{H}_M$. If $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal set in $\mathscr{H}_M$, then*

$$\langle C_r(\mathscr{P})e_1 \wedge \cdots \wedge e_r, e_1 \wedge \cdots \wedge e_r\rangle_r = [C_r(\mathscr{P}), C_r(\mathscr{M})] = E_r(\mathscr{P}\mathscr{M}), \tag{2.12}$$

where $\mathscr{M}$ is the (orthogonal projection onto the) subspace spanned by the orthonormal set $\{e_1, e_2, \ldots, e_r\}$.

Proof. We note that a property of the dyad operator $(x \times y): z \to \langle z, y\rangle x$ for x, y, z in Hilbert space $\mathscr{H}$ is that $\operatorname{tr}(x \times y) = \langle x, y\rangle$. Since, for any linear operator $\mathbf{T}$ on $\mathscr{H}$, $\mathbf{T} \cdot (x \times y) = (\mathbf{T}x \times y)$, we conclude that

$$\begin{aligned}
\langle \mathbf{T}x, y\rangle &= \operatorname{tr}((\mathbf{T}x \times y)) \\
&= \operatorname{tr}(\mathbf{T} \cdot (x \times y)) \\
&= \operatorname{tr}((y \times x)^*\mathbf{T}) \qquad (\text{since } (x \times y)^* = (y \times x)) \\
&= [\mathbf{T}, (y \times x)] \qquad (\text{from } (2.11)).
\end{aligned} \tag{2.13}$$

In this sequence of equalities, set $\mathbf{T} = C_r(\mathscr{P})$ (an operator on the Hilbert space $\wedge^q\mathscr{H}_M$), and put $x = y = e_1 \wedge e_2 \wedge \cdots \wedge e_r$. Thus (2 13) becomes

$$\begin{aligned}\langle C_r(\mathscr{P})e_1 \wedge \cdots \wedge e_r, e_1 \wedge \cdots \wedge e_r\rangle &= [C_r(\mathscr{P}), (e_1 \wedge \cdots \wedge e_r \times e_1 \wedge \cdots \wedge e_r)] \\ &= [C_r(\mathscr{P}), C_r(\mathscr{M})] \qquad (2.14)\end{aligned}$$

from (2.10) of Lemma 2.2, where $\mathscr{M}$ is the orthogonal projection onto the subspace with orthonormal basis $\{e_1, e_2, \ldots, e_r\}$. This proves one of the equalities of the lemma.

We continue (2.14) by writing

$$\begin{aligned}[C_r(\mathscr{P}), C_r(\mathscr{M})] &= \operatorname{tr}(C_r(\mathscr{P})C_r(\mathscr{M})^*) && \text{(see (2.11))} \\ &= \operatorname{tr}(C_r(\mathscr{P})C_r(\mathscr{M}^*)) \\ &= \operatorname{tr}(C_r(\mathscr{P})C_r(\mathscr{M})) && \text{(since } \mathscr{M} \text{ is orthogonal)} \\ &= \operatorname{tr}(C_r(\mathscr{P}\mathscr{M})) && \text{(since } C_r(\mathscr{P})C_r(\mathscr{M}) = C_r(\mathscr{P}\mathscr{M})) \\ &= E_r(\mathscr{P}\mathscr{M}) && \text{(see (2.5a)).} \qquad (2.15)\end{aligned}$$

The proof of the lemma is complete.

3. THE MAIN RESULT

We are in a position to prove our principal theorem.

THEOREM 3.1. *Let $P = (A_{ij})_{ij}$ be the $mn \times mn$ positive semidefinite matrix, whose ijth entry is the $n \times n$ matrix A_{ij}, $i, j = 1, 2, \ldots, m$. If P is the matrix of the positive semidefinite operator $\mathscr{P}$ relative to the orthonormal basis $\{e_1{}^1, \ldots, e_n{}^1, e_1{}^2, \ldots, e_n{}^2, \ldots, e_1{}^m, \ldots, e_n{}^m\}$, of Hilbert space $\mathscr{H}_{mn}$, then*

$$\sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} E_{pq}(P \cdot M^\tau_{\sigma^1,\ldots,\sigma^p}) \leqslant E_p \begin{bmatrix} E_q(A_{11}) & \ldots & E_q(A_{1m}) \\ & \vdots & \\ E_q(A_{m1}) & \ldots & E_q(A_{mm}) \end{bmatrix} \qquad (3.1)$$

for all elementary symmetric functions E_p, E_q, $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant n$; $M^\tau_{\sigma^1,\ldots,\sigma^p}$ denotes the matrix of the orthogonal projection $\mathscr{M}^\tau_{\sigma^1,\ldots,\sigma^p}$ onto the pq-dimensional subspace of $\mathscr{H}_{mn}$ spanned by the orthonormal set

$$\{e^{\tau(1)}_{\sigma^1(1)}, \ldots, e^{\tau(1)}_{\sigma^1(q)}, e^{\tau(2)}_{\sigma^2(1)}, \ldots, e^{\tau(2)}_{\sigma^2(q)}, \ldots, e^{\tau(p)}_{\sigma^p(1)}, \ldots, e^{\tau(p)}_{\sigma^p(1)}\},$$

where $\tau \in Q_{m,p}$ *and* $\sigma^1, \sigma^2, \ldots, \sigma^p$ *are (not necessarily distinct) functions in* $Q_{n,q}$.

Proof. We first point out that, if $P = (A_{ij})_{ij}$ is the $mn \times mn$ matrix of the positive definite linear operator $\mathscr{P}$, relative to the ordered orthonormal basis

$$\mathscr{E} = \{e_1{}^1, \ldots, e_n{}^1, e_1{}^2, \ldots, e_n{}^2, \ldots, e_1{}^m, \ldots, e_n{}^m\},$$

then the kl-th entry of the $n \times n$ block matrix A_{ij} is the scalar $\langle \mathscr{P}e_l{}^j, e_k{}^i\rangle$, i.e., $A_{ij} = (\langle \mathscr{P}e_l{}^j, e_k{}^i\rangle)_{kl}$, $k, l = 1, 2, \ldots, n$. This can be seen from the expansion

$$\mathscr{P}e_l{}^j = \sum_{s=1}^{n} \sum_{r=1}^{m} \langle \mathscr{P}e_l{}^j, e_s{}^r\rangle e_s{}^r$$

of $\mathscr{P}e_l{}^j$ relative to the orthonormal basis $\mathscr{E}$. Equivalently, we may write

$$\begin{aligned} A_{ij} = (\langle \mathscr{P}e_l{}^j, e_k{}^i\rangle)_{kl} &= (\langle \mathscr{P}^{1/2}e_l{}^j, \mathscr{P}^{1/2}e_k{}^i\rangle)_{kl} \\ &= (\langle x_l{}^j, x_k{}^i\rangle)_{kl}, \end{aligned}$$

where $x_l{}^j = \mathscr{P}^{1/2}(e_l{}^j)$; $\mathscr{P}^{1/2}$ is the unique positive semidefinite square root of $\mathscr{P}$. Since the elementary symmetric functions are transpose invariant, we may suppose the following, without loss in generality:

> Let P be the $mn \times mn$ matrix of the linear operator $\mathscr{P}$ relative to orthonormal basis $\mathscr{E}$. Let P be partitioned so that the ijth block of P is the $n \times n$ matrix A_{ij}. Then the kl-th entry of A_{ij} is the scalar $\langle x_k{}^i, x_l{}^j\rangle$, where $x_k{}^i = \mathscr{P}^{1/2}e_k{}^i$. (3.2)

With statement (3.2) in hand, we may set about finding a value for $E_q(A_{ij})$, the qth elementary function on the $n \times n$ matrix A_{ij}. For this purpose we need only apply (2.6) after setting $x_k = x_k{}^i$, $y_k = x_k{}^j$, $k = 1, 2, \ldots, n$, to obtain

$$E_q(A_{ij}) = \sum_{\sigma \in Q_{n,q}} \langle \mathbf{x}_\sigma{}^i, \mathbf{x}_\sigma{}^j\rangle_q. \tag{3.3}$$

Now that $E_q(A_{ij}) = \sum_{\sigma\in Q_{n,q}} \langle \mathbf{x}_\sigma{}^i, \mathbf{x}_\sigma{}^j\rangle_q$, we are committed to evaluate E_p on the $m \times m$ matrix

$$\begin{bmatrix} \Sigma_\sigma\langle \mathbf{x}_\sigma{}^1, \mathbf{x}_\sigma{}^1\rangle_q & \cdots & \Sigma_\sigma\langle \mathbf{x}_\sigma{}^1, \mathbf{x}_\sigma{}^m\rangle_q \\ & \vdots & \\ \Sigma_\sigma\langle \mathbf{x}_\sigma{}^m, \mathbf{x}_\sigma{}^1\rangle_q & \cdots & \Sigma_\sigma\langle \mathbf{x}_\sigma{}^m, \mathbf{x}_\sigma{}^m\rangle_q \end{bmatrix}, \tag{3.4}$$

where, in each of the m^2 sums, σ runs over $Q_{n,q}$. Now the value of the pth elementary symmetric function E_p on this $m \times m$ matrix may be evaluated by summing the determinants of all the $\binom{m}{p}$ principal $p \times p$ submatrices

$$\begin{bmatrix} \Sigma_\sigma\langle \mathbf{x}_\sigma{}^{\tau(1)}, \mathbf{x}_\sigma{}^{\tau(1)}\rangle_q & \cdots & \Sigma_\sigma\langle \mathbf{x}_\sigma{}^{\tau(1)}, \mathbf{x}_\sigma{}^{\tau(p)}\rangle_q \\ & \vdots & \\ \Sigma_\sigma\langle \mathbf{x}_\sigma{}^{\tau(p)}, \mathbf{x}_\sigma{}^{\tau(1)}\rangle_q & \cdots & \Sigma_\sigma\langle \mathbf{x}_\sigma{}^{\tau(p)}, \mathbf{x}_\sigma{}^{\tau(p)}\rangle_q \end{bmatrix} \tag{3.5}$$

of (3.4), where $\tau \in Q_{m,p}$, and σ runs over the set $Q_{n,q}$ (see (2.5c) and (2.6)). Observe that in (3.5) the m^2 summations may proceed independently of each other. But for convenience we shall require the σ's in respective columns to *agree*; we are, in effect, treating (3.4) and (3.5) as a sum of matrices over the columns. This allows us to use the multilinearity of the determinant function (relative to columns) to obtain

$$E_p \begin{bmatrix} \Sigma_{\sigma^1}\langle \mathbf{x}^1_{\sigma^1}, \mathbf{x}^1_{\sigma^1}\rangle_q & \cdots & \Sigma_{\sigma^m}\langle \mathbf{x}^1_{\sigma^m}, \mathbf{x}^m_{\sigma^m}\rangle_q \\ & \vdots & \\ \Sigma_{\sigma^1}\langle \mathbf{x}^m_{\sigma^1}, \mathbf{x}^1_{\sigma^1}\rangle_q & \cdots & \Sigma_{\sigma^m}\langle \mathbf{x}^m_{\sigma^m}, \mathbf{x}^m_{\sigma^m}\rangle_q \end{bmatrix} \tag{3.6a}$$

$$= \sum_{\tau\in Q_{m,p}} \det \begin{bmatrix} \Sigma_{\sigma^1}\langle \mathbf{x}^{\tau(1)}_{\sigma^1}, \mathbf{x}^{\tau(1)}_{\sigma^1}\rangle_q & \cdots & \Sigma_{\sigma^p}\langle \mathbf{x}^{\tau(1)}_{\sigma^p}, \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle_q \\ & \vdots & \\ \Sigma_{\sigma^1}\langle \mathbf{x}^{\tau(p)}_{\sigma^1}, \mathbf{x}^{\tau(1)}_{\sigma^1}\rangle_q & \cdots & \Sigma_{\sigma^p}\langle \mathbf{x}^{\tau(p)}_{\sigma^p}, \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle_q \end{bmatrix} \quad \text{(see (2.5c))} \tag{3.6b}$$

$$= \sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} \det \begin{bmatrix} \langle \mathbf{x}^{\tau(1)}_{\sigma^1}, \mathbf{x}^{\tau(1)}_{\sigma^1}\rangle_q & \cdots & \langle \mathbf{x}^{\tau(1)}_{\sigma^p}, \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle_q \\ & \vdots & \\ \langle \mathbf{x}^{\tau(p)}_{\sigma^1}, \mathbf{x}^{\tau(1)}_{\sigma^1}\rangle_q & \cdots & \langle \mathbf{x}^{\tau(p)}_{\sigma^p}, \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle_q \end{bmatrix}$$

$$= \sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} \langle\langle \mathbf{x}^{\tau(1)}_{\sigma^1} \barwedge \mathbf{x}^{\tau(2)}_{\sigma^2} \barwedge \cdots \barwedge \mathbf{x}^{\tau(p)}_{\sigma^p}, \mathbf{x}^{\tau(1)}_{\sigma^1} \barwedge \mathbf{x}^{\tau(2)}_{\sigma^2} \barwedge \cdots \barwedge \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle\rangle_p \tag{3.6c}$$

$$\geqslant \sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} \langle \mathbf{x}^{\tau(1)}_{\sigma^1}\wedge \mathbf{x}^{\tau(2)}_{\sigma^2}\wedge\cdots\wedge \mathbf{x}^{\tau(p)}_{\sigma^p}, \mathbf{x}^{\tau(1)}_{\sigma^1}\wedge \mathbf{x}^{\tau(2)}_{\sigma^2}\wedge\cdots\wedge \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle_{pq}$$

(from (2.9), Lemma 2.1) (3.6d))

$$= \sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} \langle C_{pq}(\mathscr{P})\mathbf{e}^{\tau(1)}_{\sigma^1}\wedge\cdots\wedge \mathbf{e}^{\tau(p)}_{\sigma^p}, \mathbf{e}^{\tau(1)}_{\sigma^1}\wedge\cdots\wedge \mathbf{e}^{\tau(p)}_{\sigma^p}\rangle_{pq}$$

(since $x_j{}^i = \mathscr{P}^{1/2}e_j{}^i$; see also (2.4)) (3.6e)

We use Lemma 2.3; set $r = pq$, and let the orthonormal set consist of the pq vectors

$$(*) \quad \{e^{\tau(1)}_{\sigma^1(1)},\ldots,e^{\tau(1)}_{\sigma^1(q)},\ldots,e^{\tau(p)}_{\sigma^p(1)},\ldots,e^{\tau(p)}_{\sigma^p(q)}\}.$$

We continue our equalities with (3.6e) equal to

$$\sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} E_{pq}(\mathscr{P}\cdot\mathscr{M}^{\tau}_{\sigma^1,\ldots,\sigma^p}), \tag{3.6f}$$

where $\mathscr{M}^{\tau}_{\sigma^1,\ldots,\sigma^p}$ is the orthogonal projection onto the space with basis (*). Thus we have shown that, for positive semidefinite matrix $P = (A_{ij})_{ij}$,

$$E_p((E_q(A_{ij})))_{ij} \geqslant \sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} E_{pq}(\mathscr{P}\cdot\mathscr{M}^{\tau}_{\sigma^1,\ldots,\sigma^p}). \tag{3.7a}$$

If we express (3.7a) relative to the induced matrices involved, we have

$$E_p(E_q(A_{ij})) \geqslant \sum_{\tau\in Q_{m,p}} \sum_{\sigma^1,\ldots,\sigma^p\in Q_{n,q}} E_{pq}(PM^{\tau}_{\sigma^1,\ldots,\sigma^p}), \tag{3.7b}$$

where, relative to the ordered, orthonormal basis $\mathscr{E}$, $P = (A_{ij})_{ij}$ is the matrix of positive definite $\mathscr{P}$, and $M^{\tau}_{\sigma^1,\ldots,\sigma^p}$ is the matrix of the orthogonal projection onto the pq-dimensional subspace spanned by the orthonormal subset (*) of $\mathscr{E}$.

As for equality for (3.7a) and (3.7b), we see that inequality occurs only in (3.6d). But equality of $\langle\langle \mathbf{x}^{\tau(1)}_{\sigma^1}\,\overline{\wedge}\cdots\overline{\wedge}\,\mathbf{x}^{\tau(p)}_{\sigma^p}, \mathbf{x}^{\tau(1)}_{\sigma^1}\,\overline{\wedge}\cdots\overline{\wedge}\,\mathbf{x}^{\tau(p)}_{\sigma^p}\rangle\rangle_p$ and $\langle \mathbf{x}^{\tau(1)}_{\sigma^1}\wedge\cdots\wedge \mathbf{x}^{\tau(p)}_{\sigma^p}, \mathbf{x}^{\tau(1)}_{\sigma^1}\wedge\cdots\wedge \mathbf{x}^{\tau(p)}_{\sigma^p}\rangle_{pq}$ is equivalent to the fact that the subspaces

$$\mathscr{M}^1 = \operatorname{sp}(x_1{}^1, x_2{}^1,\ldots,x_n{}^1), \qquad \mathscr{M}^2 = \operatorname{sp}(x_1{}^2, x_2{}^2,\ldots,x_n{}^2),\ldots,$$

$$\mathscr{M}^m = \operatorname{sp}(x_1{}^m, x_2{}^m,\ldots,x_n{}^m)$$

are elementwise orthogonal in the sense that $\langle x_k^i, x_l^j \rangle = 0$ whenever $i \neq j$, where $k, l = 1, 2, \ldots, n$. This elementwise orthogonality is equivalent to saying that the partitioned matrix P is of block diagonal form:

$$P = \begin{bmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{22} & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & A_{mm} \end{bmatrix};$$

this follows from (3.2). The theorem is proved.

COROLLARY 3.1. (R. C. Thompson [6].) *For the $mn \times mn$ positive definite matrix P, whose ijth block is the $n \times n$ matrix A_{ij}, $i, j = 1, 2, \ldots, m$, we have*

$$\det(P) = \det \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ & & \vdots & \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix}$$

$$\leqslant \det \begin{bmatrix} \det(A_{11}) & \det(A_{12}) & \ldots & \det(A_{1m}) \\ \det(A_{21}) & \det(A_{22}) & \ldots & \det(A_{2m}) \\ & & \vdots & \\ \det(A_{m1}) & \det(A_{m2}) & \ldots & \det(A_{mm}) \end{bmatrix}.$$

Equality obtains if and only if P is of block diagonal form, i.e., if P is of the form

$$P = \begin{bmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{22} & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & A_{mm} \end{bmatrix}.$$

Proof. In the statement of Theorem 3.1, set $p = m$, $q = n$. Observe then that $\tau \in Q_{m,m}$ becomes the identity function on $\{1, 2, \ldots, m\}$, and $\sigma^1 = \sigma^2 = \cdots = \sigma^p$, the only function of $Q_{n,n}$, is the identity function on $\{1, 2, \ldots, n\}$. Hence $\mathscr{M}^{\tau}_{\sigma^1, \ldots, \sigma^p}$ is the identity on $\mathscr{H}_{mn}$. This implies that the sum on the left-hand side of (3.1) reduces to the single term $E_{mn}(P) = \det(P)$. This proves the corollary.

4. A MATRICIAL INTERPRETATION FOR THEOREM 3.1

It is our intention to employ the "concrete" definition of E_q, the qth elementary symmetric function as defined on square matrices (cf. (1.3), (2.5c); see [4, p. 17]). That is, for the $n \times n$ matrix A, $E_q(A)$ is the $\binom{n}{q}$-term sum of the determinants of all the principal $q \times q$ submatrices of A. We shall recast Theorem 3.1 in this setting; then we shall be able to see why the inequality

$$E_{pq}((A_{ij})_{ij}) \leqslant E_p(E_q(A_{ij})_{ij}) \tag{4.1}$$

is not valid in general unless $p = q = 1$, or $p = m$ and $q = n$. (As before, $A = (A_{ij})_{ij}$, $ij = 1, 2, \ldots, m$, is an $m \times m$ matrix with $n \times n$ matrix block entries A_{ij}; as an $mn \times mn$ scalar-entried matrix, A is psd.)

At the heart of the matter lies a judicious means of selecting certain $pq \times pq$ principal submatrices from the $mn \times mn$ block partitioned matrix A. Thus,

Definition 4.1. We shall call a $pq \times pq$ principal submatrix, A', a *uniform principal submatrix* of A (relative to the block partitioning of A) if A' is obtained from A by extracting exactly p rows (and corresponding columns) from each of q block rows (and corresponding block columns) of A. All other principal $pq \times pq$ submatrices of A are *nonuniform*.

A schematic representation is given in (1.6) for selecting a uniform $pq \times pq$ submatrix of the $mn \times mn$ matrix A; A is partitioned into m block rows and columns, each block being an $n \times n$ matrix.

Proposition 4.1. Let A be an $mn \times mn$ matrix which is block-partitioned according to the representation $A = (A_{ij})_{ij}$, $i, j = 1, 2, \ldots, m$, where each A_{ij} is an $n \times n$ matrix. Then $E_{pq}(A)$ equals the sum of the determinants of all $pq \times pq$ uniform principal submatrices (upsm) of A + the sum of the determinants of all $pq \times pq$ nonuniform principal submatrices (nupsm) of A. Symbolically,

$$E_{pq}(A) = \sum \det(\text{upsm}) + \sum \det(\text{nupsm}). \tag{4.2}$$

Proof. The proposition is nothing more than a restatement of the fact that $E_{pq}(A)$ equals the sum of the determinants of *all* $pq \times pq$ principal submatrices of A (see (1.3) and (2.5c)).

We come now to the matric version of Theorem 3.1. Notations will be as in Proposition 4.1.

THEOREM 4.1 (Theorem 3.1). *Let $A = (A_{ij})_{ij}$, $i, j = 1, 2, \ldots, m$ be an $mn \times mn$ block-partitioned positive semidefinite matrix (each A_{ij} is an $n \times n$ matrix, $n > 1$). Then for all integers p and q, where $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant n$, we have (summing over all uniform principal submatrices),*

$$\sum \det(\text{upsm}) \leqslant E_p \begin{bmatrix} E_q(A_{11}) & E_q(A_{12}) & \ldots & E_q(A_{1m}) \\ E_q(A_{21}) & E_q(A_{22}) & \ldots & E_q(A_{2m}) \\ & & \vdots & \\ E_q(A_{m1}) & E_q(A_{m2}) & \ldots & E_q(A_{mm}) \end{bmatrix},$$

where all uniform principal submatrices are $pq \times pq$. Equality obtains if and only if A is block diagonal, i.e., $A_{ij} = 0$ whenever $i \neq j$, or when $p = q = 1$, i.e., $E_p = E_q = E_{pq} =$ trace.

Proof. Since, for any orthogonal projection $\mathcal{M}$, $\mathcal{M}^2 = \mathcal{M}$ and since, for any linear operators P and Q on mn-dimensional unitary space $\mathcal{H}_{mn}$, $E_{pq}(PQ) = E_{pq}(QP)$, we have

$$E_{pq}(P\mathcal{M}) = E_{pq}(P\mathcal{M}\mathcal{M}) = E_{pq}(\mathcal{M}P\mathcal{M}).$$

From (3.1) in Theorem 3.1, we simply write M for the pq-dimensional projection $M^{\tau}_{\sigma^1,\ldots,\sigma^p}$, and we suppress indices to obtain the equivalent statement for (3.1);

$$\sum \sum E_{pq}(PM) = \sum \sum E_{pq}(MPM) \leqslant E_p(E_q(A_{ij})_{ij}). \tag{4.3}$$

We observe that, relative to the o.n. basis $\mathcal{E}$ (see statement of Theorem 3.1), the $mn \times mn$ matrix $M^{\tau}_{\sigma^1,\ldots,\sigma^p}$ is that projection matrix which has zeros everywhere except for pq ones on certain main diagonal entries; viz., to construct $M^{\tau}_{\sigma^1,\ldots,\sigma^p}$, $\tau \in Q_{m,p}$, $\sigma^1,\ldots,\sigma^p \in Q_{n,q}$, imagine an $mn \times mn$ matrix partitioned as an $m \times m$ block matrix, each block entry being an $n \times n$ matrix. Now consider only the p block columns column $\tau(1)$, column $\tau(2),\ldots,$ column $\tau(p)$. In block column $\tau(1)$, select the q columns $\sigma^1(1), \sigma^1(2),\ldots,\sigma^1(q)$, in block column $\tau(2)$, select the q columns $\sigma^2(1)$, $\sigma^2(2),\ldots,\sigma^2(q)$, and so on, until in block column $\tau(p)$ we have selected the q columns $\sigma^p(1), \sigma^p(2),\ldots,\sigma^p(q)$. On the main diagonal of these pq columns we place a one; everywhere else we place a zero.

From this construction, it follows that, for any $mn \times mn$ matrix P, the matrix

$$(M^{\tau}_{\sigma^1,\ldots,\sigma^p} \quad P \quad M^{\tau}_{\sigma^1,\ldots,\sigma^p}) \tag{4.4}$$

is obtained from P by setting each entry equal to zero *except* for the $pq \times pq$ *uniform* principal submatrix of P, which lies in block columns $\tau(1), \tau(2),\ldots,\tau(p)$, the q columns from block column $\tau(i)$ being chosen as columns $\sigma^i(1), \sigma^i(2),\ldots,\sigma^i(q)$, $i = 1, 2,\ldots,p$. Hence E_{pq} of the matrix (4.4) coincides with the determinant of (4.4) as a $pq \times pq$ *matrix!* That is, (4.3) becomes

$$\sum_{\tau}\sum_{\tau^1,\ldots,\tau^p} \det(M^{\tau}_{\sigma^1,\ldots,\sigma^p} P M^{\tau}_{\sigma^1,\ldots,\sigma^p}) \leqslant E_p(E_q(A_{ij})_{ij}).$$

We note that the left-hand side of this inequality represents the sum of the determinants of *all* possible *uniform* principal $pq \times pq$ submatrices of $P = (A_{ij})_{ij}$. Thus (4.3) finally assumes the form

$$\sum \det(\text{upsm}) \leqslant E_p(E_q(A_{ij})_{ij}), \tag{4.5}$$

where the sum is taken over all $pq \times pq$ uniform principal submatrices of the positive semidefinite partitioned matrix $P = (A_{ij})_{ij}$, $i, j = 1, 2,\ldots, m$, where each A_{ij} is an $n \times n$ matrix. The proof is done.

Remark. We note that, if p and q are maximal, i.e., $p = m$, $q = n$, then $E_p = E_q = \det$. There is then only *one* uniform $mn \times mn$ principal submatrix of $P = (A_{ij})_{ij}$ in this case, viz., P itself; here (4.5) reduces to Thompson's result [6]

$$\det(A_{ij})_{ij} \leqslant \det(\det(A_{ij})_{ij}).$$

At the other extreme let p and q assume their minimal values, $p = q = 1$. In this case the 1×1 uniform principal submatrices are the mn main diagonal *scalars*. Thus the left-hand side of (4.5) becomes the *sum* of the (determinants of) diagonal scalars of P, i.e., the trace. We obtain then, for $P = (A_{ij})_{ij}$,

$$\operatorname{tr}(P) \leqslant \operatorname{tr}(\operatorname{tr}(A_{ij})).$$

As we have noted, equality always obtains for this case.

REFERENCES

1 J. de Pillis, Grassmann algebras as Hilbert space, *J. of Algebra* **10**(1968), 485–500.
2 J. de Pillis, Transformations on partitioned matrices, *Duke Math. J.* **36**(1969), 511–515.
3 M. Marcus, *The Use of Multilinear Algebra for Proving Matrix Inequalities*, Recent Advances in Matrix Theory, Univ. of Wisconsin Press, Madison, Wisconsin, 1964
4 M. Marcus and H. Minc, A Survey of Matrix Theory and Matrix Inequalities, Allyn and Bacon, Boston, 1964.
5 G. Mostow, J. Sampson, and J. P. Meyer, *Fundamental Structures of Algebra*, McGraw-Hill, New York, 1963.
6 R. C. Thompson, A determinantal inequality for positive definite matrices, *Canad. Math. Bull.* 4(1961), 57–62.

Received January, 1968